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A GENERALIZATION OF MAXWELL'S DEFINITION OF SOLID HARMONICS TO WAVES IN n DIMENSIONS

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Résumé

L'auteur montre qu'une expression d'un potentiel utilisant des fonctions sphériques, et donnée par Maxwell pour le cas de l'espace à trois dimensions, peut se généraliser pour l'espace à n dimensions et pour l'équation des ondes au moyen d'une formule d'un type analogue, utilisant de plus des fonctions de Bessel et de Gegenbauer.

§ 1. J. C. Maxwell in his classical „Treatise on Electricity and Magnetism" [1]) defined the potential function

$$u_n = r^{-(n+1)} P_n(\cos\theta), \tag{1}$$

where $\cos\theta = z/r$, by an n-fold differentiation in the direction of the z axis of the potential r^{-1} due to a unit charge. Thus Maxwell showed that

$$\frac{(-1)^n}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right) = \frac{1}{r^{n+1}} P_n(\cos\theta). \tag{2}$$

On the other hand a solution corresponding to (2) in the case of the wave equation

$$(\Delta + k^2)\, v = 0$$

is:

$$v_n = i^n \zeta_n(kr) \,.\, P_n(\cos\theta), \tag{3}$$

where

$$\zeta_n(kr) = \sqrt{\frac{\pi}{2kr}} \,.\, H^{(1)}_{n+\frac{1}{2}}(kr), \tag{4}$$

and in particular:

$$\zeta_o(kr) = \frac{e^{ikr}}{ikr}, \tag{4a}$$

$H^{(1)}$ being the Hankel-function of the first kind.

Lord Rayleigh in his, also classical, „Theory of Sound" [2]), referring to Maxwell's proof of (2) remarks: „It might perhaps have been expected that a similar law would hold for the velocity potential with the substitution of $r^{-1}\, e^{ikr}$ for r^{-1}. This however is not the case."

It is the purpose of the present note

a) to show that in fact (3) can be obtained by differentiation of (4), although the law for this wave solution is more complicated than the corresponding law for the potential solution (2),

b) to extend the result to more dimensions. Incidentally we obtain

c) some new relations between various functions occurring in the analysis.

§ 2. Considering (3) we remark with Professor E. T. Whittaker [3]) that when v_n is a wave solution, also will be $\partial v_n/\partial z$. Therefore differentiation of (3) with respect to z must yield again a wave solution. In fact, we find, with

$$\frac{\partial}{\partial z} = \cos\theta \, . \, \frac{\partial}{\partial r} - \frac{1}{r} \sin\theta \, . \, \frac{\partial}{\partial \theta} \, ,$$

$$i^{-n} \frac{\partial v_n}{\partial (kz)} = \zeta_n'(kr) \, . \cos\theta \, . \, P_n(\cos\theta) - \frac{1}{kr} \zeta_n(kr) \, . \sin\theta \, . \, P_n'(\cos\theta). \tag{5}$$

But by the two recurrence relations between Bessel-functions and spherical harmonics:

$$\zeta_n' = -\frac{n+1}{kr}\zeta_n + \zeta_{n-1}$$

and

$$\sin\theta \, . \, P_n' = \frac{n(n+1)}{2n+1}(P_{n+1} - P_{n-1}),$$

(5) can be reduced to

$$(2n+1)\frac{\partial v_n}{i\partial(kz)} = (n+1)\, v_{n+1} + n v_{n-1}\, ; \tag{6}$$

thus (6) embodies in one equation both the above recurrence relations.

Further, (6) is seen to be of the same form as the well known recurrence relation

$$(2n+1)\, \mu \, . \, P_n(\mu) = (n+1)\, P_{n+1}(\mu) + n P_{n-1}(\mu) \tag{7}$$

between spherical harmonics, only with μ replaced by $\partial/i\partial(kz)$. Hence it is natural to try the solution

$$v_n \equiv i^n \zeta_n(kr) \,.\, P_n(\cos\theta) = P_n\left(\frac{\partial}{i\partial(kz)}\right)(v_o) = P_n\left(\frac{\partial}{i\partial(kz)}\right)\left(\frac{e^{ikr}}{ikr}\right) \qquad (8)$$

which, on test, is found confirmed. It is natural to consider the P functions as the solution of (7) because the other solution (the Q functions) cannot be developed in positive powers of their argument, which here is the differential symbol $\partial/i\partial(kz)$.

Hence (8) shows that the wave solutions v_n of order *n can* be obtained by differentiation of the elementary solution $v_o = e^{ikr}/ikr$ for a point source, and again in the direction of the z axis, although not in as simple a way as Maxwell obtained the potential functions (2). It is of interest to remark that in (8) the differentiation prescribed by the operator $P_n(\partial/i\partial(kz))$ again yields the result, as in (2), in the form of a product of a function of r only and a function of θ only.

Further taking in (8) the special case $x = y = 0$, i.e. $\theta = 0$ and therefore $P_n(1) = 1$, we obtain:

$$i^n \,.\, \zeta_n(kz) = P_n\left(\frac{\partial}{i\partial(kz)}\right) \cdot \left(\frac{e^{ikz}}{ikz}\right), \qquad (9)$$

a result found by Lord Rayleigh [4]).

§ 3. Returning now to two dimensions, we consider the wave solutions

$$w_n = i^n J_n(k\rho) \,.\, \cos n\varphi,$$

with $x = \rho\cos\varphi$, $y = \rho\sin\varphi$.

Operating as above, i.e. differentiating in the x-direction and applying again a recurrence relation of the Bessel-functions, we find

$$2\,\frac{\partial w_n}{i\partial(kx)} = w_{n+1} + w_{n-1},$$

which is of the same form as a recurrence relation of the Tchebycheff polynomia [5]), which are defined as:

$$T_n(\mu) = \tfrac{1}{2}\{(\mu + i\sqrt{1-\mu^2})^n + (\mu - i\sqrt{1-\mu^2})^n\} = \cos n\varphi,$$

where $\mu = \cos\varphi$.

Hence, as in the three-dimensional case, we obtain the relation, analogous to (8):

$$T_n\left(\frac{\partial}{i\partial(kx)}\right)(J_o(k\rho)) \equiv \cos\left\{n\cos^{-1}\left(\frac{\partial}{i\partial(kx)}\right)\right\}(J_o(k\rho)) = i^n J_n(k\rho).\cos n\varphi \quad (10)$$

and again the operation of $T_n(\partial/i\partial kx)$ on $J_o(k\rho)$ yields a result in separated form.

The special case $y = \varphi = 0$ yields

$$T_n\left(\frac{\partial}{i\partial(kx)}\right)(J_o(kx)) = i^n . J_n(kx). \quad (11)$$

§ 4. Wave solutions similar to those obtained above but in $2\nu + 2$ dimensions are known to be

$$u_n = i^n \frac{Z_{n+\nu}(kr)}{(kr)^\nu} . C_n^\nu(\cos\theta),$$

where $Z_{n+\nu}$ is any Bessel-function of order $n + \nu$,

$$\cos\theta = x_n/r$$

$$r^2 = \sum_1^n x_i^2,$$

and where the Gegenbauer-functions $C_n^\nu(\mu)$ are defined by the more dimensional potential functions

$$\left(1 - 2\mu\frac{r}{a} + \frac{r^2}{a^2}\right)^{-\nu} = \sum_{n=0}^{n=\infty}\left(\frac{r}{a}\right)^n . C_n^\nu(\mu). \qquad \begin{matrix}(r < a), \\ (\nu \geqslant \frac{1}{2}).\end{matrix}$$

A procedure similar to the above yields the general expression:

$$C_n^\nu\left(\frac{\partial}{i\partial(kx_n)}\right)\left(\frac{Z_\nu(kr)}{(kr)^\nu}\right) = i^n \frac{Z_{n+\nu}(kr)}{(kr)^\nu} . C_n^\nu(\cos\theta), \quad (12)$$

where again the operation gives a result in a separated form. The special case $\theta = 0$, viz.:

$$C_n^\nu\left(\frac{\partial}{i\partial x}\right)\left(\frac{J_\nu(x)}{x^\nu}\right) = i^n \frac{J_{n+\nu}(x)}{x^\nu} . \frac{(2\nu + n - 1)!}{n!\,(2\nu - 1)!}$$

was already obtained by Watson [6]).

Finally (12) shows that in all the above expressions, where a Bessel-function or Hankel-function occurs, it may be replaced in both members by any solution of Bessel's equation of the same order.

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4) l. c. page 263.
5) For a summary of their properties, see e. g. Balth. van der Pol, Physica **1**, 78, 1933.
6) Besselfunctions, Cambridge 1922, p. 50.